Analytic Function

① diff ⇐ لو قابلة للتفاضل عند z_0.

② Analytic ⇐ " " " " " وجوارها.

③ entire ⇐ " " " عند جميع نقط Z-plane.

* أى دالة تحتوى على $\bar{z}$ أو $|z|$ غير analytic.

* الدالة (analytic) لو حققت شروط Cauchy-Rieman

$$\boxed{\begin{aligned} U_x &= V_y \\ U_y &= -V_x \end{aligned}} \qquad \boxed{\begin{aligned} U_r &= \frac{1}{r} V_\theta \\ V_r &= \frac{-1}{r} U_\theta \end{aligned}}$$

$$f'(z) = U_x + i V_x \qquad f'(z) = e^{-i\theta}(U_r + i V_r)$$

⑤ * Prove that $\frac{d}{dz}(z^2\bar{z})$ doesn't exist any where.

Sol

$f(z) = z^2 \bar{z} = (x+iy)^2 (x-iy)$

$= (x^2 + i2xy - y^2)(x-iy)$

$= x^3 + i2x^2y - y^2x - ix^2y + 2xy^2 + iy^3$

$= \underbrace{(x^3 + y^2x)}_{u} + i\underbrace{(x^2y + y^3)}_{v}$

$u_x = 3x^2 + y^2$ $\quad\quad$ $u_y = 2xy$

$v_x = 2xy$ $\quad\quad$ $v_y = x^2 + 3y^2$

$u_x \neq v_y \quad , \quad u_y = -v_x$

not analytic $\Rightarrow$ not diff. $\neq$

$\boxed{6}$ Are the following function analytic ?

a) $f(z) = Re\,(z^2) = Re\,((x+iy)^2)$

$= Re\,(x^2 + i2xy - y^2)$

$f(z) = x^2 - y^2$

$u = x^2 - y^2$ $\quad\quad$ $v = 0$

$u_x = 2x$ $\quad\quad$ $u_y = -2y$

$v_x = 0$ $\quad\quad$ $v_y = 0$

$u_x \neq v_y \,,\; u_y \neq -v_x$

$\longrightarrow$ not analytic

b) $f(z) = z - \bar{z} = (x+iy) - x + iy$

$= i2y$

$u = 0 \qquad u_x = 0 \qquad u_y = 0$

$v = 2y \qquad v_x = 0 \qquad v_y = 2$

$\longrightarrow$ not analytic

ⓒ $f(z) = z^2 = (x+iy)^2$

$= x^2 + i2xy - y^2$

$u = x^2 - y^2 \qquad v = 2xy$

$u_x = 2x \qquad u_y = -2y$

$v_x = 2y \qquad v_y = 2x$

$u_x = v_y$, $u_y = -v_x \Longrightarrow$ analytic

d) $f(z) = \frac{x}{x^2+y^2} + i \frac{y}{x^2+y^2}$

$u = \frac{\cancel{x}\,y}{x^2+y^2}$, $v = \frac{\cancel{y}\,x}{x^2+y^2}$

$\cancel{u_x = \frac{(x^2+y^2) - x\,2x^2}{(x^2+y^2)^2}}$

$u_x = \frac{-2xy}{(x^2+y^2)^2}$ $\qquad u_y = \frac{x^2-y^2}{(x^2+y^2)^2}$

$v_x = \frac{-x^2+y^2}{(x^2+y^2)}$, $\qquad v_y = \frac{-2xy}{(x^2+y^2)^2}$

$u_x = v_y$

$u_y = -v_x$ $\Bigg\rangle$ analytic

* Let $f(z) = u + iv$ analytic fn

$u = 3x^2y + 2x^2 - y^3 - 2y^2$ find $f'(z)$

without $f(z)$ and Conjugate harmonic (v)

<u>Sol</u>

$$f'(z) = u_x + iV_x = u_x - iu_y$$

$$= (6xy + 4x) - i(3x^2 - 3y^2 - 4y)$$

$$u_x = V_y = 6xy + 4x$$

$$V_x = -u_y = -3x^2 + 3y^2 + 4y$$

$$V = \int V_y = 3xy^2 + 4xy + f(x)$$

هنفاضل V بالنسبة
ل x وساويناها ب V_x

$$V_x = 3y^2 + 4y + f'(x) = -3x^2 + 3y^2 + 4y$$

$$f'(x) = -3x^2 \implies f(x) = -x^3 + c$$

$$V = 3xy^2 + 4xy + \cancel{f(x)} - x^3 + c$$

Harmonic fn

$U_{xx} + U_{yy} = 0 \quad , V_{xx} + V_{yy} = 0$

$r^2 U_{rr} + r U_r + U_{\theta\theta} = 0$

كل (analytic) تعتبر
harmonic

* Prove that $u = \ln(x^2 + y^2)$ harmonic.

$$u_x = \frac{2x}{x^2 + y^2} \longrightarrow u_{xx} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$u_y = \frac{2y}{x^2 + y^2} \longrightarrow u_{yy} = \frac{-2y^2 + 2x^2}{(x^2 + y^2)^2}$$

$u_{xx} + u_{yy} = 0 \longrightarrow$ harmonic

* determine which the following fn are Possible to be real Part of analytic fn or not.

ⓐ $u = x^2 + y + 3$

$u_x = 2x$ $\qquad$ $u_{xx} = 2$

$u_y = 1$ $\qquad$ $u_{yy} = 0$

$u_{xx} + u_{yy} \neq 0$

→ This Function is not harmonic ⇒ not analytic

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b) $u = \cosh x \cosh y$

$u_x = \sinh x \cosh y$ $\qquad$ $u_{xx} = \cosh x \cosh y$

$u_y = \cosh x \sinh y$ $\qquad$ $u_{yy} = \cosh x \cosh y$

$u_{xx} \mp u_{yy} \neq 0$

→ not harmonic ⇒ not analytic

* show that if $f(z) = u + iv$ is analytic

$$\nabla^2 |f(z)|^2 = 4 \left| \frac{df}{dz} \right|^2 \qquad \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

$u^2 + v^2$

$$\left|\frac{df}{dz}\right|^2 = u_x^2 + V_x^2$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial}{\partial x}(u^2+v^2) = 2uu_x + 2VV_x$$

$$\frac{\partial^2}{\partial x^2}(u^2+v^2) = 2uu_{xx} + 2u_x^2 + 2VV_{xx} + 2V_x^2 \rightarrow (1)$$

$$\frac{\partial^2}{\partial y^2}(u^2+v^2) = 2uu_{yy} + 2uu_y^2 + 2VV_{yy} + 2V_y^2 \rightarrow (2)$$

~~from (1)~~

(1) + (2)

~~2u (u_xx~~

$$= 2u(u_{xx} + u_{yy}) + 2(u_x^2 + V_x^2)$$

$$+ 2u(V_{xx} + V_{yy}) + 2(u_y^2 + V_y^2)$$

$\cancel{u_{xx}+u}$

$$u_{xx}+u_{yy}=0$$

$$V_{xx}+V_{yy}=0$$

$$u_x^2+V_x^2=f'(z)$$

$$u_y^2+V_y^2=f'(z)$$

$\therefore$ ① + ②

$$=4\left|\frac{df}{dz}\right|^2$$

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